WNE Linear Algebra
Final Exam
Series A

2 February 2023

Questions

Question 1.
Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism given by the formula

$$\varphi((x_1, x_2)) = (4x_1 + tx_2, x_1 + 2x_2).$$

For which $t \in \mathbb{R}$ is the vector $v = (1, 1)$ an eigenvector of φ? Find the corresponding eigenvalue.

Solution 1.

$$\varphi((1, 1)) = (4 + t, 3) = \lambda(1, 1),$$

if and only if $\lambda = 3$ and $4 + t = 3$, i.e. $t = -1$.

Question 2.
Let $A \in M(n \times n; \mathbb{R})$ be a diagonalizable matrix. If $C^{-1}AC = D$ is a diagonal matrix for some invertible matrix $C \in M(n \times n; \mathbb{R})$, does it follow that columns of $(C^\intercal)^{-1}$ are eigenvectors of the matrix $A^\intercal$?

Solution 2.
Yes, it does. If $C^{-1}AC = D$ then $C^\intercal A^\intercal \left(C^{-1}\right)^\intercal = D^\intercal$ and $(C^\intercal)^{-1} = \left(C^{-1}\right)^\intercal$.

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ is an antisymmetric matrix, i.e. $A^\intercal = -A$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unit matrix, does it follow that matrix $A - I$ is invertible?

Solution 3.
Matrix A is antisymmetric if and only if $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, for some $a \in \mathbb{R}$. Hence

$$\det(A - I) = 1 + a^2 > 0,$$

which implies that $A - I$ is invertible.

Question 4.
Matrix $P = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$ is a matrix of an orthogonal projection onto some subspace $V \subset \mathbb{R}^2$. Find an orthonormal basis of $V^\perp$.

Solution 4.
The subspace V is equal to the image of P, i.e. $\mathrm{lin}((1, 2))$ (it is spanned by columns of P). The subspace $V^\perp$ is equal to the kernel of P, which is orthogonal to V, i.e. $\mathrm{lin}((2, -1))$. An orthonormal basis of $V^\perp$ is, for example, $\frac{1}{\sqrt{5}}(2, -1)$.

Question 5.
A system of linear equations in two variables has two solutions $(1, 1)$ and $(1, 3)$. Give an example of a third solution different from the previous ones.

Solution 5.

The set of all solutions of a system of linear equations is an affine subspace hence it is closed under affine combinations. Another solutions is for example

$$\frac{1}{2}(1,1) + \frac{1}{2}(1,3) = (1,2).$$

In fact, for any $b \in \mathbb{R}$ vector $(1, b)$ is a solution of that system.